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LARGE DEFLECTIONS OF AXISYMMETRIC CIRCULAR MEMBRANES†

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Abstract—A nonlinear relaxation method is employed to solve the nonlinear partial differential equations governing the large deflection response of various axisymmetric circular membranes. The method proposed here is an iterative approach used in conjunction with finite difference approximations and in its simplest form consists of only two operators. In principle, this method offers a technique of systematically reducing the errors at each nodal point for each algebraic equation to some acceptable level. In addition, the method, simple in logic but powerful in application, is believed to be applicable to solve general types of nonlinear equations. The problems solved herein include uniformly loaded circular membrane, annular membrane with rigid central disc and annular membrane with free inner edge. The numerical results obtained in this paper compare quite well with other results given in the literature. Moreover, many of the results obtained here may be readily used in practical engineering design.

INTRODUCTION

IN ORDER to obtain a realistic distribution of stresses of membrane-like structures, large deflection (nonlinear) theory is essential. The deformation of initially flat circular membrane, described by Foppl–Hencky large deflection theory [1–3], has been the subject of numerous investigations. Hencky [4] and Chein [5] determined the deflection and stress fields for solid membrane. Goldberg and Pifko employed both an iterative technique [6] and a power series approach [7] to obtain the solutions for annular membranes with various inner boundary conditions. Iberall [8] gave a tractable solution of the circular membrane with a floating rigid central disc by using certain simplifying approximation. Sherbourne and Lennox [9] re-examined the same central disc problem by employing a fourth-order Runge–Kutta process and found that their result differed appreciably from Iberall's by almost a factor of two. The reason for this discrepancy is unexplained.

A nonlinear finite difference relaxation method discussed previously [10] is introduced herein to solve a number of axisymmetric nonlinear membrane problems, i.e. (a) solid circular membrane, (b) annular membrane with free inner edge and (c) annular membrane with central disc. The solution method employed here has been successfully utilized to solve some very complicated shell problems [11, 12] and appears to be extremely effective relative to other approaches [13, 14].

The present solution method enjoys the following advantages: (a) simplicity of logic that makes it a trivial task to learn how to employ it, (b) versatility and ease of application,

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(c) insensitivity to starting values as far as convergence is concerned and (d) less computer storage required relative to other methods especially when one deals with higher order partial differential equations.

In the next section the governing equations will be enumerated. This will be followed by a brief description of solution method. In the subsequent sections three different cases of membrane problems will be examined. In the final section, a discussion and conclusions will be presented.

BASIC EQUATIONS

Consider a circular membrane with axisymmetric geometry and loading as shown in Fig. 1. The strain–displacement relations in polar coordinates are

$$\varepsilon_r = \bar{u}_{,r} + \tfrac{1}{2}\bar{w}_{,r}^2 \tag{1a}$$

$$\varepsilon_\theta = \frac{\bar{u}}{r} \tag{1b}$$

where $\bar{u}$ and $\bar{w}$ are displacement components along radial and normal directions, respectively, and $(\ \ )_{,r} = (\mathrm{d}/\mathrm{d}r)(\ \ )$.

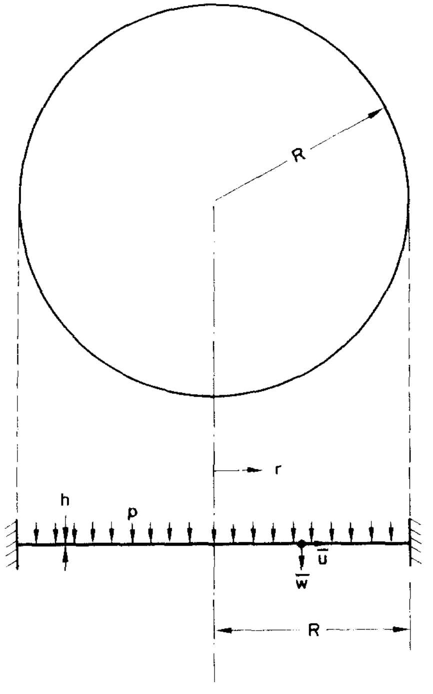

FIG. 1. Geometry and notation for a uniformly loaded membrane.

The stress resultants in radial and circumferential directions, N_r and N_θ, are given in terms of displacements by

$$N_r = \frac{Eh}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) = \frac{Eh}{1-\nu^2}\left[\bar{u}_{,r} + \frac{1}{2}\bar{w}_{,r}^2 + \nu\frac{\bar{u}}{r}\right] \tag{2a}$$

$$N_\theta = \frac{Eh}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) = \frac{Eh}{1-\nu^2}\left[\frac{\bar{u}}{r} + \nu\bar{u}_{,r} + \frac{\nu}{2}\bar{w}_{,r}^2\right] \tag{2b}$$

where E, ν are material constants and h is the thickness of the membrane.

The equation enforcing equilibrium in the radial direction is

$$(rN_r)_{,r} - N_\theta = 0 \tag{3}$$

while the governing equilibrium equation in the transverse direction is as follows:

$$\frac{1}{r}(rN_r\bar{w}_{,r})_{,r} + p = 0. \tag{4}$$

Substitution of equations (2) in equations (3) and (4) yields the following two equilibrium equations in terms of displacements:

$$3\bar{w}_{,rr}\bar{w}_{,r}^2 + \frac{\bar{w}_{,r}^3}{r} + 2\bar{w}_{,rr}\left(\bar{u}_{,r} + \nu\frac{\bar{u}}{r}\right) + 2\bar{w}_{,r}\left[\bar{u}_{,rr} + (1+\nu)\frac{\bar{u}_{,r}}{r}\right] + \frac{2(1-\nu^2)p}{Eh} = 0 \tag{5}$$

$$\bar{u}_{,rr} + \frac{\bar{u}_{,r}}{r} - \frac{\bar{u}}{r^2} + \bar{w}_{,r}\bar{w}_{,rr} + \frac{1-\nu}{2}\frac{\bar{w}_{,r}^2}{r} = 0. \tag{6}$$

A convenient nondimensional form follows if we write

$$\begin{aligned} x &= r/R, \qquad \bar{w} = wR(pR/Eh)^{1/3} \\ \bar{u} &= uR(pR/Eh)^{2/3}. \end{aligned} \tag{7}$$

Introducing equation (7) into equations (5) and (6) we obtain the following dimensionless equilibrium equations:

$$3w_{,xx}w_{,x}^2 + \frac{w_{,x}^3}{x} + 2w_{,xx}\left(u_{,x} + \nu\frac{u}{x}\right) + 2w_{,x}\left[u_{,xx} + (1+\nu)\frac{u_{,x}}{x}\right] + K = 0 \tag{8}$$

$$u_{,xx} + \frac{u_{,x}}{x} - \frac{u}{x^2} + w_{,x}w_{,xx} + \frac{1-\nu}{2}\frac{w_{,x}^2}{x} = 0 \tag{9}$$

where

$$K = 2(1-\nu^2), \qquad (\quad)_{,x} = \frac{\mathrm{d}}{\mathrm{d}x}(\quad).$$

Boundary conditions associated with each individual case are given in the corresponding example problem sections. Equations (8) and (9) together with given boundary conditions are the basic equations to be solved by the numerical analysis.

For convenience, stress components, $\bar{\sigma}r$ and $\bar{\sigma}_\theta$, are obtained from equations (2) and put into dimensionless forms as follows:

$$\bar{\sigma}_r = N_r/h = \sigma_r(p^2R^2E/h^2)^{1/3} \tag{10a}$$

$$\bar{\sigma}_\theta = N_\theta/h = \sigma_\theta(p^2R^2E/h^2)^{1/3} \tag{10b}$$

where

$$\sigma_r = \frac{1}{1-\nu^2}\left(u_{,x}+\frac{1}{2}w_{,x}^2+\nu\frac{u}{x}\right) \tag{10c}$$

$$\sigma_\theta = \frac{1}{1-\nu^2}\left(\frac{u}{x}+\nu u_{,x}+\frac{\nu}{2}w_{,x}^2\right). \tag{10d}$$

The essence of the problem before us is to calculate w, σ_r and σ_θ from the solutions of equations (8) and (9) with associated proper boundary conditions.

METHOD OF SOLUTION

The basic concept and iteration procedure of the nonlinear relaxation method has been described in fair detail in Ref. [10] in which three very different, difficult problems were used to demonstrate its variety of application.

Here only a brief discussion of the method will be given. In order to effectively illustrate the method procedure, it is considered advisable to use a simple one-dimensional problem as a model example, while recognizing that there should be no more conceptual difficulty when it is applied to more complex problems.

Let us consider the following nonlinear differential equation for the domain shown in Fig. 2

$$(y'')^2 = c. \tag{11}$$

With replacement of the second derivative at generic point i by its finite difference approximation, we have

$$\left(\frac{y_{i+1}-2y_i+y_{i-1}}{a^2}\right)^2 - c = 0$$

where a is the mesh spacing and c is a constant.

A so-called residual operator, R_i, is introduced by simply replacing the zero term on the right hand side of the last equation by

$$R_i = \left(\frac{y_{i+1}-2y_i+y_{i-1}}{a^2}\right)^2 - c. \tag{12}$$

The problem is considered to be solved when, for a given set of values of y_i, the residuals or "error terms" R_i at all nodes are zero or acceptably small. Thus, a procedure must be

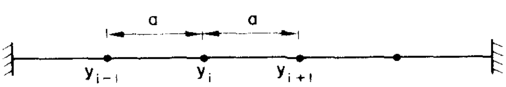

FIG. 2. One-dimensional mesh for equation $(y'')^2 = c$.

developed to systematically liquidate all R_i. To this end, another important operator, the relaxation operator, is necessary and is derived as follows:

$$\frac{\partial R_i}{\partial y_i} = -\frac{4}{a^4}(y_{i+1} - 2y_i + y_{i+1}). \tag{13}$$

This operator, which is in fact approximate for nonlinear equations, is used to calculate the total amount of change of function, Δy_i, necessary to bring R_i toward zero. Therefore

$$\Delta y_i = -R_i/(\partial R_i/\partial y_i). \tag{14a}$$

Consequently, the improved value of y_i is obtained by

$$(\text{new})\, y_i = (\text{old})\, y_i + \Delta y_i. \tag{15}$$

The operation can then move to the next point, say $i + 1$, by introducing this new y_i whenever it is called upon. Such a point to point operation can be set up systematically from say left to right. After a complete sweep throughout whole domain, an examination of a convergence criterion which is introduced is advisable.

Two obvious possibilities as convergence criteria are to require that either the residuals (R_i) or the percentage change of the function ($\Delta y_i/y_i$) are acceptably small at all points. Still another criterion which because of its efficiency is here adopted is required that the average absolute change of the function is quite small as follows:

$$\sum_{i=1}^{N} \left|\frac{\Delta y_i}{y_i}\right| \leq 0.0005 \tag{16}$$

where N is the total number of all nodal points. When condition (16) is satisfied the iteration process is halted and the solution printed out.

In order to accelerate convergence, a so-called over-relaxation factor is frequently introduced in equation (14a) and takes the form

$$\Delta y_i = -\omega R_i/(\partial R_i/\partial y_i) \tag{14b}$$

in which ω may take a value of between 1 and 2. Typically, a value of ω of 1·3 cuts computer time in half.

For simultaneous equations such as equations (8) and (9) considered herein, the solution procedure is followed in substantially the same manner, requiring only a minor modification.

First, equation (8) is considered to be a function of w only by viewing the u function (which may have a different value from one node to another) as temporarily fixed; the iteration procedure described above can be readily applied. Next, in equation (9), we take the w function at all nodes as temporarily fixed; the equation would accordingly be thought of as function of u and the same iteration procedure is again applied. Therefore, in the solution process, these steps are taken in turn with one set of equations and then the other, so that we are dealing alternatively with a nonlinear system of equations.

The iteration process is continued until both w and u functions at all nodes simultaneously satisfy the criterion of (16).

Uniformly loaded solid circular membrane with fixed edge

The first problem considered is one for which a classical solution exists, viz. the uniformly loaded circular membrane with fixed peripheral edge (Fig. 3). Because of radial symmetry, the problem domain is confined to a generic radius with boundary conditions as follows:

$$x = 0, \qquad u = w,_x = 0 \quad \text{(center)} \tag{17a}$$

$$x = 1, \qquad u = w = 0 \quad \text{(clamped edge)} \tag{17b}$$

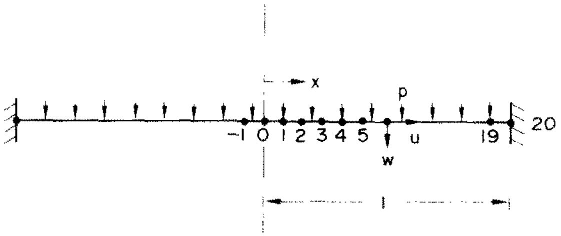

FIG. 3. Finite difference mesh along a generic radius of a uniformly loaded solid circular membrane with fixed edge.

The problem at hand is to determine the deflection functions w and u which simultaneously satisfy equations (8) and (9) in the domain x between 0 and 1. The solution is obtained by the nonlinear relaxation method as outlined in the Method of Solution section. When the average absolute values of percentage changes of the deflections at all mesh points throughout the domain are less than 0·05 per cent for both u and w functions, the solutions are said to have converged. In numerical treatment, the domain x between 0 and 1 (Fig. 3) is divided into twenty evenly spaced mesh points; these are numbered in order from 0 to 20.

Because of symmetry, the central point ($x = 0$) must be considered carefully. Clearly, $w,_x$ and u are zero at the center so that equation (8), the equilibrium equation, is reduced appreciably. On the other hand, $u,_x$ is not zero but in fact takes on its largest value there. After obtaining the value of w at all points by sweeping through the field and relaxing each point in turn, we obtain w_0 by simply substituting in the aforementioned reduced form of equation (8).

Alternatively, we could obtain w_0 by passing a high order polynomial through a number of mesh points adjacent to the center. This latter technique was used here with a fifth degree polynomial.

A comparison of the present central deflection and central and edge radial stresses with Hencky's classical (but slighly corrected [5]) solution is shown in Table 1.

The computing time required to obtain the solution starting with guessed values that were of the order of 200 per cent above the correct solution was 10 sec of CPU time on a GE 635 time-shared computer.

Uniformly loaded annular membrane fixed at outer edge and with a rigid central disc

The configuration of an annular membrane is shown in Fig. 4, consisting of a flexible annulus connected to a rigid central disc and fixed along the outer edge. The boundary

TABLE 1

	w_0 Central deflection	σ_{r0} Central stress	$\sigma_{r\,edge}$ Edge stress
Hencky original [4]	0·666	0·423	0·328
Hencky corrected [5]	0·6536	0·431	—
Present solution	0·6541	0·4289	0·3306

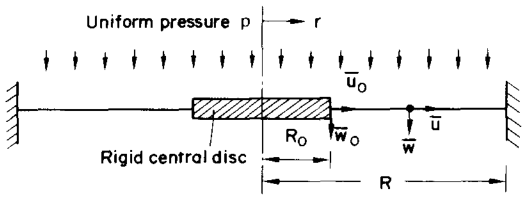

FIG. 4. Geometry of a uniformly loaded annular membrane fixed at the outer edge and with a central rigid disc.

conditions at the outer edge are the same as those in the previous example, i.e.

$$\text{at} \quad x = 1, \qquad w = u = 0. \tag{18}$$

Next, if the central disc is taken as a free body, then the boundary condition at the connection between membrane and central disc can be derived by considering the vertical equilibrium of the forces acting on this free body. Thus we have

$$p\pi R_0^2 + 2\pi R_0(\bar{\sigma}_r h)\frac{dw}{dr} = 0. \tag{19}$$

Replacing the radial stress ($\bar{\sigma}_r = Nr/h$) by displacement components [equation (2a)] and introducing nondimensional quantities given in equation (7) and (10), we obtain

$$\text{at} \quad x = x_0, \qquad \left(\frac{1}{2}w_{,x}^2 + u_{,x} + v\frac{u}{x}\right)w_{,x} + \frac{1}{4}Kx_0 = 0 \tag{20a}$$

where K has been defined in equation (9), and $x_0 = R_0/R$. If it is further assumed that, at the edge x_0 the disc and membrane are rigidly attached, then we have

$$\text{at} \quad x = x_0, \qquad u = 0. \tag{20b}$$

Thus, governing equations (8) and (9) together with boundary equations (18) and (20) complete the formulation of the title problem.

Equation (20a) is the so-called mixed type nonlinear boundary condition in which two dependent variables (u and w) along with the nonlinear terms are presented. In the theoretical or numerical analysis, the solution required to satisfy this condition is indeed a formidable task. However, in the u problem [equation (9)], the u function at $x = x_0$, according to equation (20b), vanishes (i.e. $u_0 = 0$) and the function at the other points in the domain can be obtained by utilizing a simple relaxation procedure.

On the other hand, in the w problem w_0 must satisfy equation (20a) and an attempt to the solution of w_0 results in solving a nonlinear problem. To this end, we number the w function from 0 towards the right by $w_0, w_1, w_2, \ldots$, and replace $(w_{,x})_0$ by a five point fifth order polynomial.

$$(w_{,x})_0 = \frac{1}{2a}(-3w_0+3w_1+2w_2-3w_3+w_4) \tag{21}$$

wherein a again stands for mesh spacing.

We wish to calculate the value of w_0 which satisfies boundary condition (20a). Substituting equation (21) into (20a) and introducing a residual term we obtain the following finite difference equivalent of (20a):

$$\begin{aligned} R_{w_0} &= \frac{1}{2a}(-3w_0+3w_1+2w_2-3w_3+w_4) \\ &\quad\times\left[\frac{1}{8a^2}(-3w_0+3w_1+2w_2-3w_3+w_4)^2+S\right]+\frac{1}{4}Kx_0 \end{aligned} \tag{22a}$$

where

$$S = (u_{,x})_0 + v\frac{u_0}{x_0}.$$

Equation (22a) is a residual operator and its corresponding relaxation operator can be straightforwardly obtained as follows:

$$\frac{\partial R_{w_0}}{\partial w_0} = -\frac{9}{16a^3}(-3w_0+3w_1+2w_2-3w_3+w_4)^2-3S/2. \tag{22b}$$

With these two operators set up, a simple relaxation procedure can then be utilized. The relaxation iteration should be repeated until the desired accuracy for w_0 is achieved. Subsequently, we return to the field points to solve the finite difference representation of equation (8).

In this problem, no "exact" previous solutions exist and only two approximate solutions [8, 9] are available. Sherbourne and Lennox [9] integrated the differential equations by a fourth order Runge–Kutta process and obtained a solution, $w_0 = 0{\cdot}292$,† associated with a particular geometry $x_0 = 0{\cdot}7$ ($v = \frac{1}{3}$). However, Iberall [8] utilized integro-differential equations with certain simplified approximations and obtained $w_0 = 0{\cdot}166$† for the same geometry. Iberall's result differed from Sherbourne and Lennox's solution by almost a factor of two. For the same geometry condition, we obtain a value of w_0 equal to 0·2708 which is quite close to Sherbourne and Lennox's. Naturally, this good correlation between the present result and Sherbourne and Lennox's does cast a shadow of doubt on Iberall's solution.

The maximum stress for the case considered here occurs at the connection between membrane and central disc ($r = R_0$) in the direction along radius. The value of this maximum stress divided by central stress of solid circular membrane is plotted in Fig. 5 for x_0 ($=R_0/R$) ranging from 0 to 0·8. Clearly, the most undesirable situation occurs when the

† The values of w reported here are the average values obtained from an interpretation of results given in Fig. 3 of Ref. [9].

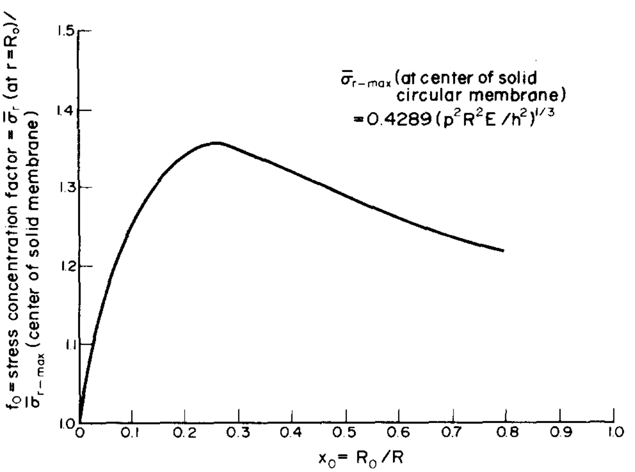

FIG. 5. Stress concentration factors of the circular membrane with rigid central disc (Fig. 4), $\nu = 0{\cdot}3$.

disc radius is about one-quarter the outer radius; the associated stress increase over the solid membrane case is almost 40 per cent.

Uniformly loaded annular membrane fixed at outer edge and free from tractions at the inner edge

The configuration of this problem is sketched in Fig. 6 in which boundary conditions at both edges are as follows:

$$\text{at } x = 1 \qquad w = u = 0 \quad \text{(fixed edge)} \tag{23a}$$

$$\text{at} \quad x = x_0 \qquad \frac{1}{2}w_{,x}^2 + u_{,x} + \nu\frac{u}{x} = 0 \quad \text{(free edge).} \tag{23b}$$

Once again we encounter a mixed type boundary condition, equation (23b), which is a little simpler than that treated in the previous example. The same procedure as outlined in the previous case can also be employed to obtain the solution of this problem.

A comparison of the present normal deflections with the power series solutions† obtained by Pifko and Goldberg [7] is shown in Fig. 7. Undoubtedly, these two solutions are in excellent agreement. Here, it should be pointed out that the formulations used by

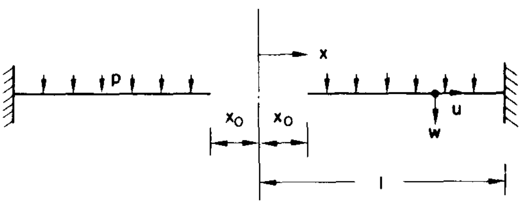

FIG. 6. Geometry of a uniformly loaded annular membrane fixed at outer edge and free at the inner edge.

† A positive truncation error was introduced in each case of Pifko and Goldberg's w solutions, which resulted from approximating the solution by neglecting the higher order terms in the power series expansion. Therefore the results he obtained are the lower bound solutions. The present results are consistently above his results.

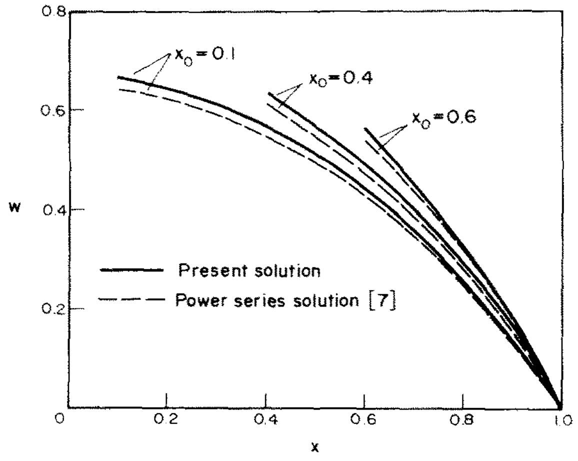

FIG. 7. Comparison of normal deflections for the annular membrane shown in Fig. 6 with power series solutions in Ref. [7] ($\nu = 0{\cdot}3$).

Pifko and Goldberg differ appreciably from the one used here and additionally, the two solution methods, as one might expect, are completely different. This evidence suggests the validity of the differential equations derived herein, the computer program developed and methodology of the nonlinear relaxation technique utilized.

In practical engineering design, the stress concentration factor at the free edge is usually of interest. Therefore, the definition of this factor as well as its value for different x_0 ranging from 0·1 to 0·6 are given and shown in Fig. 8. These results are presented in a convenient form for immediate engineering utility.

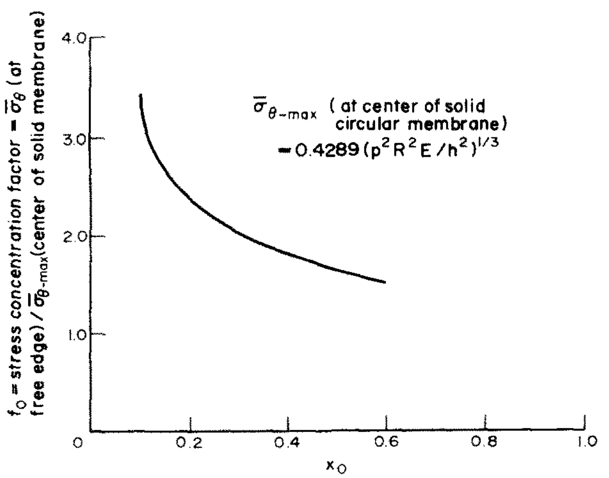

FIG. 8. Stress concentration factors of the membrane shown in Fig. 6 ($\nu = 0{\cdot}3$).

As can be seen from the figure the stress concentration factor as a result of the circular hole is very significant, reaching a value of about $3\frac{1}{2}$ when the hole radius is $\frac{1}{10}$ the outside radius.

CONCLUSIONS

A nonlinear relaxation method used in conjunction with finite difference approximations is applied to solve the geometrically nonlinear circular membrane equations. The method utilized here is similar to the one used by Shaw and Perrone [15] except for the fact that carry-over factors were utilized in the earlier analysis. In their approach, the carry-over factors were used to calculate approximate induced residuals in the neighborhood of the point operated. The derivation of these factors becomes very tedius for more complex problems (for example, see Ref. [15]) and their removal should greatly enhance the usefulness of the method.

In principle, the current method offers a technique of systematically reducing the errors at each nodal point for each algebraic equation to some acceptable level by means of residual and relaxation operators.

The essential advantages of this method are: (a) versatility and ease of application, (b) simplicity of logic that makes it a trivial task to learn how to employ it, (c) accuracy of the method which is limited only by errors associated with finite difference approximations, (d) insensitivity to starting value as far as convergence is concerned, at least for the problems considered here and (e) less computer storage required relative to other methods especially when one deals with higher order partial differential equations.

In order to increase the speed of convergence, an over-relaxation factor is introduced by which the computing time is cut down to about one-half of the original, a factor of about 1·3 appears to be a judicious choice.

In this paper, three example problems of axisymmetric circular membranes are treated: (a) clamped solid circular membrane, (b) clamped annular membrane with a floating central disc and (c) clamped annular membrane with free inner edge. For the solid membrane, excellent correlation is obtained between the present solution and Hencky's corrected solution [5].

In the second example problem, the annular membrane with a floating central disc, we encounter a mixed type nonlinear boundary condition at the juncture of the membrane and disc. Solution to this type of problem is a formidable task and accordingly, a special treatment at the inner edge of the membrane is necessary; the boundary condition is treated as a nonlinear problem and nonlinear relaxation method is utilized to obtain the deflection at this edge. For this problem no "exact" solution exists and only two approximate solutions [8, 9] are available. The present solution is quite close to Sherbourne and Lennox's [9] while differing from Iberall's [8] by almost a factor of two.

In the third example, a membrane with a "hole", we encounter again a mixed type nonlinear boundary condition. The same technique employed in the previous case is also applied here. The present solutions compare fairly well with Pifko and Goldberg's power series solutions [7] which actually are lower bound solutions. The present results are consistently above the solutions in Ref. [7]. For practical design purposes, a series of stress concentration factors at the juncture of the membrane and disc (Fig. 4), and at the free edge (Fig. 6) are plotted in Figs. 5 and 8, respectively; these figures might be readily used directly by engineers.

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Абстракт—Применяется нелинейный метод релаксации с целью решения нелинейных дифференциальных уравнений в частных производных, описывающих поведение разных, осесимметрических, круглых мембран при больших прогибах. Предлагаемый здесь метод является итерационным способом, использованным одновременно с аппроксимацией в конечных разностях. В своей найболее простой форме этот метод состоит из двух операторов. В припципе, метод дает способ систематически ограничевающий погрешности в каждои узловой точке для каждого алгебраического уравнения, на некотором допускаемом уровне. Кроме того, простой по логике но опличный с точки зрения применяемости, метод можно употребнить для расчета общих типов нелинейных уравнений. Задачи, решены здесь, касаются равномерно нагруженной круглой мембраны, кольцевой мембраны с жестким, центральным диском и кольцевой мембраны со свободным внутренным краем. Полученные в работе, численные результаты сравниваются совсем акуратно с другими результатами известными в литературе .Далее, большинство результатов, решеных в работе, можно легко использовать в инженерной практике.